SPHERICALLY SYMMETRIC VACUUM SOLUTIONS IN FIRST ORDER GRAVITY: FATE OF CURVATURE SINGULARITIES

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- GENERAL RELATIVITY IS FAMOUSLY KNOWN TO EXHIBIT <u>BLACK HOLE</u> SOLUTIONS

- SUCH CONFIGURATIONS ARE ASSOCIATED WITH CURVATURE <u>SINGULARITIES</u>

- THE SIMPLEST EXAMPLE OF A BH SOLUTION IS PROVIDED BY THE SCHWARZSCHILD METRIC: (SPHERICALLY SYMMETRIC, STATIC):

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

- THIS DESCRIBES THE SPACETIME GEOMETRY OUTSIDE A (SPH. SYM) MASS M; CHARACTERISED BY A SURFACE AT <u>$r = 2M$</u> WHICH HAS A <u>DEGENERATE</u> 3-METRIC AND DIVIDES SPACETIME INTO AN INTERIOR ($r < 2M$) AND AN EXTERIOR ($r > 2M$) REGION

- THE SCHW. METRIC CONTINUES TO BE A SOLUTION OF EINSTEIN EQ^{N} (IN VACUUM) $R_{\mu\nu} = 0$ EVEN AT THE INTERIOR

- IF ONE CONTINUES INSIDE, A CURVATURE SINGULARITY AT $r=0$ IS ENCOUNTERED, AS THE SO(3,1) FIELD STRENGTH COMPONENTS $R_{\mu\nu}{}^{IJ}(\omega)$ AS WELL AS THE SCALAR

$$R_{\mu\nu}{}^{IJ} R^{\mu\nu}{}_{IJ} = R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} = \frac{48 M^2}{r^6}$$ BLOW UP

- WHEN THE METRIC IS INVERTIBLE ($\det g_{\mu\nu} \neq 0$), THERE SEEMS TO BE NO ESCAPE FROM SUCH A SCENARIO IN GR

- POSSIBLE WAY OUT TO EVADE SINGULARITIES: MODIFIED GRAVITY, EXTRA DIMENSIONS, EXOTIC MATTER, QUANTUM EFFECTS, ...

- HOWEVER, ONE CAN STILL CHOOSE TO REMAIN CONSERVATIVE: IS IT POSSIBLE TO CONSTRUCT A WELL-BEHAVED INTERIOR GEOMETRY IN PURE GRAVITY WITHOUT MODIFYING THE STANDARD THEORY (ACTION PRINCIPLE / LAGRANGIAN)?

- IN THAT CASE, THE ONLY POSSIBLE OPTION (PERHAPS) IS TO WORK WITH DEGENERATE (NON-INVERTIBLE) METRIC AT THE INTERIOR!

VACUUM SOLUTIONS WITH DEGENERATE INTERIOR METRIC

- SUCH A SOLUTION, IF IT EXISTS, SHOULD ALSO EXHIBIT THE FOLLOWING PROPERTIES:

1. BOTH THE (NON-DEGENERATE) EXTERIOR METRIC AND THE (DEGENERATE) INTERIOR METRIC MUST BE SOLUTIONS OF VACUUM EOM IN GRAVITY THEORY

2. THE INTERIOR & EXTERIOR GEOMETRIES MUST BE JOINED SUFFICIENTLY SMOOTHLY AT THE 'HORIZON'

3. THE CONFIGURATION MUST BE 'REGULAR' EVERYWHERE

CONSTRUCTION OF EXTERIOR SOLUTION:

- LET US CONSIDER A SPH. SYM. STATIC METRIC OF THE FOLLOWING FORM (FOR THE EXTERIOR REGION):

$$ds^2 = -\left(1-\frac{2M}{f(u)}\right)dt^2 + \frac{f'^2(u)}{\left(1-\frac{2M}{f(u)}\right)}du^2 + f^2(u)\,d\Omega^2_{\theta,\phi}$$

$[-\infty < u < \infty]$

- $$f(u) = 2M\left[1-\left(1-\frac{u}{u_0}\right)^{2l+1}\right], \quad l \geq 1$$

$$f(u_0) = 2M, \quad f'(u_0) = 0$$

$$\Rightarrow \underline{\operatorname{Det} g_{\mu\nu}(u_0) = 0}$$

- THIS METRIC IS GAUGE-EQUIVALENT TO THE SCHW. METRIC AT $u > u_0$, BUT <u>NOT</u> AT $u = u_0$

- $u < u_0$, $u > u_0$, $u = u_0$ & $u = 0$ CORRESPOND TO $r < 2M$, $r > 2M$, $r = 2M$ & $r = 0$, RESPECTIVELY; HOWEVER, THE REGION $-\infty < u < 0$ HAS NO ANALOGUE IN SCHW. COORDINATES.

- AT THE 'HORIZON' $u = u_0$, THE $SO(3,1)$ FIELD-STRENGTH COMPONENTS $R_{\mu\nu}{}^{IJ} = \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J}$ BECOME:

$$R^{01} \overset{*}{=} R^{02} \overset{*}{=} R^{03} \overset{*}{=} 0 ,$$

$$R^{12} \overset{*}{=} R^{31} \overset{*}{=} 0 , \quad R^{23} \overset{*}{=} \sin\theta \, d\theta \wedge d\phi$$

- TO EMPHASIZE, THE COMPONENTS OF $g_{\mu\nu}$ and $R_{\mu\nu}{}^{IJ}$ at $u = u_0$ FOR THE EXTERIOR GEOMETRY MUST MATCH THOSE FOR THE INTERIOR GEOMETRY, WHICH WE CONSTRUCT NEXT.

CONSTRUCTION OF INTERIOR: DEGENERATE METRIC

- THE AIM IS TO CONSTRUCT AN INTERIOR ($u<u_0$) GEOMETRY THAT CONNECTS SMOOTHLY TO THE (DEGENERATE) HORIZON, AND IS BASED ON A NON-INVERTIBLE METRIC.
- THIS MUST BE A SOLUTION OF THE EOMs IN PURE GRAVITY
- THESE EOMs MUST COME FROM A LAGRANGIAN THAT CAN ACCOMODATE NON-INVERTIBLE METRICS
- HENCE, THE APPROPRIATE LAGRANGIAN (DENSITY) MUST CORRESPOND TO THE FIRST ORDER FORMULATION, WHICH DOES NOT REQUIRE THE INVERSE METRIC IN ITS CONSTRUCTION:

$$\mathcal{L}(e,\omega) = \frac{1}{K^2} \epsilon^{\mu\nu\alpha\beta} \epsilon_{IJKL}\, e_\mu{}^I e_\nu{}^J R_{\alpha\beta}{}^{KL}(\omega)$$

$$S = \frac{1}{K^2}\int d^4x \; \mathcal{L}(e,\omega)$$

[HILBERT-PALATINI]

DEGENERATE SOLUTIONS: NON-EINSTEINIAN GRAVITY

- THIS IS TO BE CONTRASTED WITH THE SECOND ORDER LAGRANGIAN (DENSITY), WHICH IS ILL-DEFINED FOR DEGENERATE METRICS, AND DOES NOT ACCOMODATE DEGENERATE SOLUTIONS:

$$\mathcal{L}(g) = \frac{1}{k^2}\sqrt{-g}\; g^{\mu\nu} R_{\mu\nu}(g)$$

- THE EOMS, COMING FROM THE VARIATIONS OF $\mathcal{L}(e,\omega)$ W.R.T. e_μ^I & ω_μ^{IJ}, ARE:

$$\delta\omega: \quad e_{[\mu}^{[I} D_\nu e_{\alpha]}^{J]} = 0$$

$$\delta e: \quad e_{[\mu}^{[I} R_{\alpha\beta]}^{JK]} = 0$$

- WHEN $\mathrm{Det}\, e_\mu^I \neq 0$, THESE EOMs CORRESPOND TO EINSTEIN'S THEORY

- WHEN $\mathrm{Det}\, e_\mu^I = 0$, ONE DOES NOT RECOVER EINSTEIN'S THEORY

- SOLUTIONS OF THE FIRST-ORDER EOM WITH DEGENERATE METRICS GENERICALLY EXHIBIT TORSION ; THESE ARE NOT PERCEIVED AT ALL BY THE SECOND-ORDER THEORY BASED ON INVERTIBLE $g_{\mu\nu}$.

  [TSEYTLIN 1982, KAUL & SENGUPTA 2016]

- HOW TO CONSTRUCT A DEGENERATE SPACETIME SOLUTION (INTERIOR) IN THE PRESENT CONTEXT OF A SPHERICALLY SYMMETRIC GEOMETRY?

  (IN ADDITION,) CAN ONE OBTAIN WELL-BEHAVED SO(3,1) FIELD-STRENGTH COMPONENTS EVERYWHERE, AND ALSO SATISFY THE REQUIREMENT OF CONTINUITY PROPERTIES AT THE 'HORIZON' $u=u_0$?

INTERIOR: AN EXPLICIT DEGENERATE SOLUTION

- FOR $u \leq u_0$, CONSIDER THE DEGENERATE METRIC ($g_{tt}=0$) WITH TOPOLOGY $\mathbb{R} \otimes S^2$:

$$ds^2_{(4)} = 0 + F^2(u)\,du^2 + H^2(u)\left[d\theta^2 + \sin^2\theta\, d\phi^2\right]$$

- TETRAD:

$$e_\mu^{\ I} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F(u) & 0 & 0 \\ 0 & 0 & H(u) & 0 \\ 0 & 0 & 0 & H(u)\sin\theta \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & e_a^{\ i} \end{pmatrix}$$

- SPIN-CONNECTION:

$$\omega^{0i} = 0, \qquad \omega^{ij} = \bar{\omega}^{ij}(e) + K^{ij}$$

$$[I \equiv (0, i) \equiv (0, 1, 2, 3)]$$

- MOST GENERAL K^{ij} SATISFYING THE EOMs: $K^{ij} = \epsilon^{ijk} e_a^{\ l} N_{kl}$ Where

$$N_{kl} = N_{lk} \equiv \begin{pmatrix} \alpha & \eta_3 & \eta_2 \\ \eta_3 & \beta & \eta_1 \\ \eta_2 & \eta_1 & \gamma \end{pmatrix}$$

- THE FIELDS $e_\mu^I \equiv (F(u), H(u))$ and $\omega_\mu^{IJ} \equiv (\bar{\omega}_\mu^{IJ}(e), K_\mu^{IJ})$ SATISFY THE 1ST ORDER EOM-S, PROVIDED THEY ARE RELATED THROUGH THE CONSTRAINT:

$$\eta_1^2+\eta_2^2+\eta_3^2-\alpha\beta-\beta\gamma-\gamma\alpha=\frac{H'^2}{F^2H^2}+\frac{2}{FH}\left(\frac{H'}{F}\right)'-\frac{1}{H^2}$$

$$\Rightarrow \quad C(\eta_1,\eta_2,\eta_3,\alpha,\beta,\gamma,F,H)=0$$

- FOR SIMPLICITY, WE CONSIDER '<u>MINIMAL</u>' TORSION: ONLY $\eta_1 \neq 0$. AND ASSUME THAT $\eta_1=\eta_1(u)$ [SPH. SYM].

- THE ABOVE CONSTRAINT BECOMES:

$$C(\eta_1, F, H)=0$$

- NEED 2 FURTHER CONDITIONS TO <u>DETERMINE</u> η_1, F & H.

INTERIOR GEOMETRY :

- THE $SO(3,1)$ FIELD-STRENGTH COMPONENTS:

$$R^{12} = \left(\eta_1 H - \frac{H'}{F}\right)' du \wedge d\theta$$

$$R^{23} = \left[1 + \left(\eta_1 H - \frac{H'}{F}\right)\left(\eta_1 H + \frac{H'}{F}\right)\right] \sin\theta \, d\theta \wedge d\phi$$

$$R^{31} = \left(\eta_1 H + \frac{H'}{F}\right)' \sin\theta \, du \wedge d\phi + 2\eta_1 H \cos\theta \, d\theta \wedge d\phi$$

$$R^{0i} = 0$$

- CONTINUITY OF $R_{\mu\nu}{}^{IJ}$ AT $u = u_0$ ("HORIZON") IMPLIES:

$$R^{12}(u_0) = 0 \;, \; R^{23}(u_0) = \sin\theta \, d\theta \wedge d\phi , \; R^{31}(u_0) = 0,$$

$$R^{0i}(u_0) = 0$$

- THE 2 CONDITIONS TO BE CHOSEN MUST BE CONSISTENT WITH THE ABOVE; WE CHOOSE:

$$\eta_1 H = \frac{H'}{F} \;,$$

$$\frac{H'}{F} = \beta e^{\alpha(u-u_0)}(u_0 - u)^m \;, \quad m > 1$$

- THESE IMPLY: $\overset{(INT)}{R_{\mu\nu}}{}^{IJ} = \overset{(EXT)}{R_{\mu\nu}}{}^{IJ}$ AT $u = u_0$

- THE 3 UNKNOWNS η_1, F, H CAN NOW BE SOLVED USING THE 3 EQUATIONS:

$$\eta_1 = \eta_1(u), \quad F = F(u), \quad H = H(u).$$

- $$ds^2_{INT} = 0 + F^2(u)\,du^2 + H^2(u)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$F(u) \longrightarrow 0$ as $u \longrightarrow u_0$ or $u \longrightarrow -\infty$

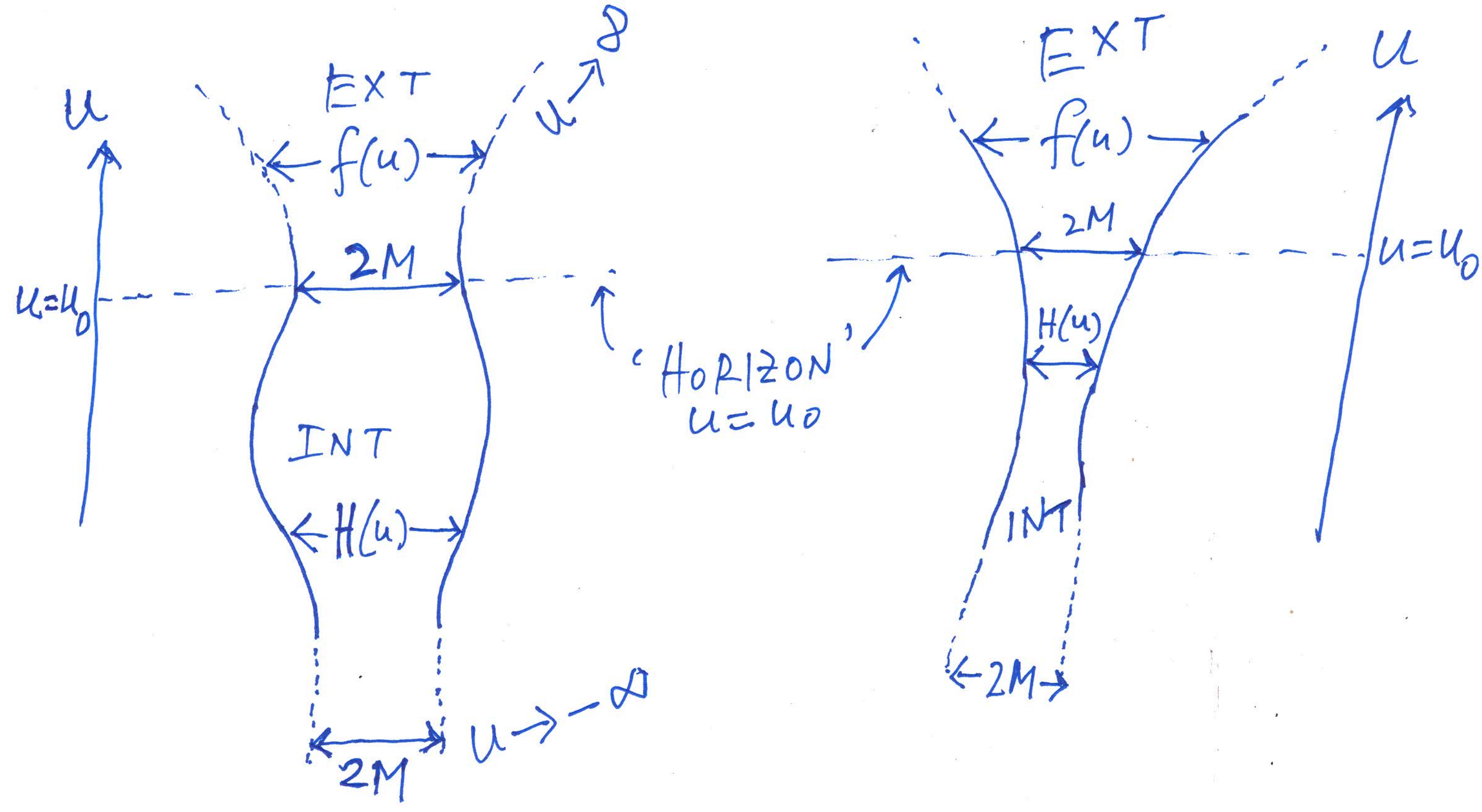
$H(u) \longrightarrow 2M$ as $u \longrightarrow u_0$ or $u \longrightarrow -\infty$

$\eta_1(u) \longrightarrow 0$ as $u \longrightarrow u_0$ or $u \longrightarrow -\infty$

- THIS, IN TURN, IMPLIES THAT: $\overset{(INT)}{g_{\mu\nu}} = \overset{(EXT)}{g_{\mu\nu}}$ AT $u = u_0$

- ALL THE $SO(3,1)$ FIELD-STRENGTH COMPONENTS $R_{\mu\nu}{}^{IJ}$ ARE FINITE AT $-\infty < u < \infty$ [UNLIKE SCHW. INTERIOR]

PICTORIAL REPRESENTATION : (INTERIOR + EXTERIOR ≡ FULL SOLUTION)



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TO SUMMARISE ....

- A NEW CLASS OF <u>VACUUM</u> SOLUTIONS (OF FIRST ORDER EOM-s) HAS BEEN CONSTRUCTED ; HAS <u>INVERTIBLE</u> METRIC AT THE <u>EXTERIOR</u> & <u>NON-INVERTIBLE</u> METRIC AT THE HORIZON AND THE <u>INTERIOR</u>

- THE SOLUTION IS 'REGULAR' EVERYWHERE, SINCE THE $SO(3,1)$ FIELD-STRENGTH COMPONENTS ARE <u>FINITE</u> EVERYWHERE

- THIS IS <u>UNLIKE</u> THE CASE OF INTERIOR SCHW. SOLUTION DESCRIBING A <u>BLACK HOLE</u>, WHERE THESE ARE ILL-DEFINED AT THE ORIGIN $r=0$.

- THE METRIC & FIELD-STRENGTH COMPONENTS ARE CONTINUOUS AT THE HORIZON

- THE DEGENERATE INTERIOR SPACETIME SOURCES TORSION

- THE ORIGIN OF TORSION IN THIS CONFIGURATION IS PURELY GEOMETRIC (NOT MATTER-COUPLING)

- THE CLASS OF SOLUTIONS CORRESPOND TO VANISHING (FINITE!) ACTION; SHOULD BE TREATED AS SADDLE POINTS IN PATH INTEGRAL QUANTUM GRAVITY

TO EXPLORE FURTHER ....

- ARE THESE SOLUTIONS REALLY NON-SINGULAR ?

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  BEHAVIOUR OF GEODESIC CONGRUENCES

- THE TORSIONAL FIELD CONTENT IS CONFINED AT THE INTERIOR; HOW TO MAKE USE OF THIS ADDITIONAL FIELD CONTENT ?

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  MANY EXCITING POSSIBILITIES

- HOW TO MODEL STARS (INTERIOR WITH MATTER) OR OTHER POSSIBLE REALISTIC SYSTEMS ?

THANK

YOU!